An Application of Splitting Methods to Convex Partially Separable Optimization Problems*

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Abstract

This paper aims to develop a numerical approach on the basis of necessary and sufficient conditions for the existence of a solution for convex partially separable optimization problems. Specially, applications of splitting methods to spline approximation problems are discussed. Numerical examples are presented.

Key words: Convex partially separable optimization problems, the Fenchel-Lagrange duality, optimality conditions, convex interpolation with cubic splines, projection algorithm.

1 Introduction

After introducing the so-called Fenchel-Lagrange dual problem by Boţ and Wanka ([11]), a lot of interesting theoretical results have been published (see for instance [3] and [12]). On the other hand, it is necessary to investigate which advantages has the new dual problem from a practical point of view. In association with this reason

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we aim to show numerical advantages for some optimization problems. For instance, after deriving the Fenchel-Lagrange dual problem for convex partially separable optimization problems, strong duality assertions and optimality conditions have been investigated (see [1]). 1990s were shown by Schmidt et al. (cf. [4], [9] and [10]) that the construction of shape preserving spline approximation problems can be reduced into the study of convex partially separable optimization problems. Although, by using the Lagrange duality they can efficiently solvable by the Newton method, there are still few works devoted to the development of new numerical algorithms. Some analysis related to using the Newton method have been investigated (see [5] and [6]).

Since the objective function for the Fenchel-Lagrange dual problem is nondifferentiable in general, differentiable optimization methods can not be applied to solve this problem. But, by reducing necessary and sufficient conditions for existence a solution into the generalized equations, it can usable the so-called splitting algorithms for finding zero of the sum of maximal monotone operators (cf. [2]). For different classes of splitting algorithms and their convergence we refer to [7].

The aim of this paper deals with the development a numerical approach on the basis of necessary and sufficient conditions for the existence of a solution for convex partially separable optimization problems.

This paper is organized as follows. In Section 2 we recall some results dealing with the Fenchel-Lagrange duality for convex partially separable optimization problems. In particular, the assertion how to reduce optimality conditions into the generalized equations is stated. The next section is devoted to suggest a numerical algorithm in order to construct convex interpolation with C^1 splines. Moreover, some numerical examples are presented.

2 The Fenchel-Lagrange dual problem for convex partially separable optimization problems

Assume that $F_i : \mathbb{R}^{l_i} \to \mathbb{R},\ i = \overline{1,n}$, are convex functions and $W_i \subseteq \mathbb{R}^{l_i},\ \ i = \overline{1,n}$, are closed, convex sets. Let $A_i \in \mathbb{R}^{l_i \times (n+1)},\ l_i \in \{1, ..., n+1\}$ be given matrices. We consider the following optimization problem

$$(P^{cps})\ \ \inf_{u \in W} \sum_{i=1}^{n} F_i(A_i u),$$

where

$$W = \left\{ u = (u_0, ..., u_n)^T \in \mathbb{R}^{n+1} \middle|\ A_i u \in W_i,\ i = \overline{1,n} \right\}.$$

Introducing the auxiliary variables $v_i = A_i u \in \mathbb{R}^{l_i},\ i = \overline{1,n}$, (P^{cps}) can be rewritten as

$$(P^{cps})\ \ \inf_{v \in V} \sum_{i=1}^{n} F_i(v_i),$$

where

$$V = \left\{ v \in \mathbb{R}^{k} \middle|\ v_i - A_i u = 0,\ v_i \in W_i,\ i = \overline{1,n} \right\},$$

with $v = (u, v_1, ..., v_n) \in \mathbb{R}^k$ and $k = n + 1 + l_1 + ... + l_n$.

In [1], for a more general case of (P^{cps}) and its particular cases different dual problems and the optimality conditions have been derived. For instance, the Fenchel-Lagrange dual problem to (P^{cps}) becomes

$$(D^{cps}_{FL}) \qquad \sup_{\substack{q_i, p_i \in \mathbb{R}^{l_i}, i = \overline{1,n} \\ \sum_{i=1}^{n} A_i^T q_i = 0}} \left\{ -\sum_{i=1}^{n} F_i^*(p_i) + \sum_{i=1}^{n} \inf_{v_i \in W_i} (p_i + q_i)^T v_i \right\}.$$

The functions F_i^* are the conjugates of F_i defined by

$$F_i^*(p_i) = \sup_{x_i \in \mathbb{R}^{l_i}} [p_i^T x_i - F_i(x_i)],\ i = \overline{1,n}.$$

The Fenchel-Lagrange dual problem has been investigated as a "combination" of the classical Lagrange and Fenchel dual problems in convex optimization (see [11]).

In association with the optimality conditions, the following assertion has been verified.

Proposition 1. *([2]) Assume that* $\exists u' \in \mathbb{R}^{n+1}$ *such that* $A_i u' \in ri(W_i),\ i = \overline{1,n}$. *Then* $\bar{u} \in W$ *is an optimal solution to* (P^{cps}) *if and only if* $\forall i \in \{1,...,n\},\ \bar{v}_i = A_i\bar{u} \in \mathbb{R}^{l_i}$ *is a solution to the following generalized variational inequality problem:* $\exists \bar{p}_i \in \partial F_i(\bar{v}_i)$ *such that*

$$(GVI^i_{cps}) \qquad (\bar{p}_i + \bar{q}_i)^T(v_i - \bar{v}_i) \geq 0,\ \forall v_i \in W_i,$$

where $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$.

Where $\mathrm{ri}(C)$ is the relative interior of a given set $C \subseteq \mathbb{R}^n$ and ∂h denotes the subdifferential of a given function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$.

$\forall i \in \{1,...,n\},\ (GVI^i_{cps})$ leads to the inclusion problems of finding $\bar{v}_i \in \mathbb{R}^{l_i}$ such that

$$(IP^i_{cps}) \qquad 0 \in \bar{q}_i + N_{W_i}(\bar{v}_i) + \partial F_i(\bar{v}_i), \tag{1}$$

where $\bar{q}_i \in \mathbb{R}^{l_i}$ fulfills $\sum\limits_{i=1}^{n} A_i^T \bar{q}_i = 0$ and N_C is the normal cone operator defined by

$$N_C(x) = \begin{cases} \{z \in \mathbb{R}^n |\ z^T(y - x) \leq 0,\ \forall y \in C\}, & \text{if } x \in C; \\ \varnothing, & \text{otherwise.} \end{cases}$$

The convex partially separable optimization problems were intensively investigated by Schmidt et al. ([9] and [10]) related to the construction shape preserving spline approximation problems. In order to construct a convex spline interpolant, it is appropriate to select such spline which has minimal curvature. This requirement leads to the following problem

$$(P^{ts}) \qquad \inf_{\substack{(u_{i-1},u_i) \in W_i \subseteq \mathbb{R}^{2s} \\ i=\overline{1,n}}} \sum_{i=1}^{n} F_i(u_{i-1}, u_i),$$

which is called the tridiagonally separable optimization problem. For the problem (P^{ts}) and for its particular cases dealing with the spline approximation problems, the Lagrange duals are unconstrained. If the solutions to the Lagrange duals are known, then the primal problems can be solved by the so-called return-formula (see [4], [9] and [10]). On the other hand, the reformulation of the optimality conditions for the Fenchel-Lagrange dual problem to (P^{ts}) becomes the inclusion problem ([2]).

The Fenchel-Lagrange dual problem to (P^{ts}) turns out to be (see [1])

$$\begin{aligned}(D^{ts}_{FL}) \qquad & \sup_{\substack{(p_{i1},p_{i2})\in\mathbb{R}^{2s}\\ q_i\in\mathbb{R}^s, i=\overline{0,n}}} \Big\{ -\sum_{i=1}^{n} F_i^*(p_{i1},p_{i2}) \\ & + \sum_{i=1}^{n} \inf_{(u_{i-1},u_i)\in W_i} [(p_{i1}-q_{i-1})^T u_{i-1} + (p_{i2}+q_i)^T u_i] \Big\} \\ & \text{with } q_0 = 0,\ q_n = 0.\end{aligned}$$

By Proposition 4.2 in [1], one can show that a point $\bar{u} = (\bar{u}_0, ..., \bar{u}_n)^T \in \underbrace{\mathbb{R}^s \times \cdots \times \mathbb{R}^s}_{n+1}$ is an optimal solution to (P^{ts}) if and only if (see (1))

$$0 \in (-\bar{q}_{i-1}, \bar{q}_i) + N_{W_i}(\bar{u}_{i-1}, \bar{u}_i) + \partial F_i(\bar{u}_{i-1}, \bar{u}_i),\ i = \overline{1, n}, \tag{2}$$

where $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, ..., \bar{p}_n) \in \underbrace{\mathbb{R}^{2s} \times \cdots \times \mathbb{R}^{2s}}_{n},\ \bar{q} = (\bar{q}_0, \bar{q}_1, ..., \bar{q}_n)$ $\in \underbrace{\mathbb{R}^s \times \cdots \times \mathbb{R}^s}_{n+1}$ is an optimal solution to (D^{ts}_{FL}) and it holds

$$\bar{p}_i = (\bar{p}_{i1}, \bar{p}_{i2}) \in \partial F(\bar{u}_{i-1}, \bar{u}_i),\ i = \overline{1, n}. \tag{3}$$

3 Computational results for convex interpolation with C^1 splines

Let $(x_i, y_i)^T \in \mathbb{R}^2,\ i = \overline{0, n}$, be a given data points defined on the grid

$$\Delta_n :\ x_0 < x_1 < ... < x_n.$$

A cubic spline s on Δ_n can be given on $[x_{i-1}, x_i]$ by

$$\begin{aligned} s(x) &= y_{i-1} + m_{i-1}(x - x_{i-1}) \\ &+ (3\tau_i - 2m_{i-1} - m_i)\frac{(x - x_{i-1})^2}{h_i} + (m_{i-1} + m_i - 2\tau_i)\frac{(x - x_{i-1})^3}{h_i^2} \end{aligned}$$

with $h_i = x_i - x_{i-1},\ \tau_i = \frac{y_i - y_{i-1}}{h_i},\ i = \overline{1,n}$.
It holds $s \in C^1[x_0, x_n]$ and $s(x_i) = y_i,\ s'(x_i) = m_i,\ i = \overline{0,n}$. The points $(x_0, y_0), ..., (x_n, y_n)$ associated to Δ_n are said to be in convex position if

$$\tau_1 \le \tau_2 \le \cdots \le \tau_n. \tag{4}$$

If (4) is fulfilled, then the convexity of s leads to the problem of finding

$$(x, y)^T \in W_i = \{(x, y)^T \in \mathbb{R}^2 |\ 2x + y \le 3\tau_i \le x + 2y\},\ i = \overline{1,n}. \tag{5}$$

In order to select only one convex interpolant will be minimized the mean curvature of s. It is easily verified that

$$\begin{aligned} &\min_{\substack{(m_{i-1}, m_i)^T \in W_i \\ i=\overline{1,n}}} \int_{x_0}^{x_n} s''(x)^2 dx \\ = &\min_{\substack{(m_{i-1}, m_i)^T \in W_i \\ i=\overline{1,n}}} \Big(\sum_{i=1}^{n} \frac{4}{h_i^2}\{m_i^2 + m_i m_{i-1} + m_{i-1}^2 - 3\tau_i(m_i + m_{i-1}) + 3\tau_i^2\} \Big), \end{aligned}$$

where the feasible set is given by (5). In other words, we obtain the tridiagonally problem with $s = 1$ and the functions are given by

$$F_i(x, y) = \frac{4}{h_i^2}\{x^2 + xy + y^2 - 3\tau_i(x + y) + 3\tau_i^2\},\ i = \overline{1,n}$$

and the sets by (5). The conjugate functions are easily calculated as

$$\begin{aligned} F_i^*(\xi, \eta) &= \sup_{x,y \in \mathbb{R}} \{x^T\xi + y^T\eta - F_i(x, y)\} \\ &= \tau_i(\xi + \eta) + \frac{h_i}{12}(\xi^2 + \eta^2 - \xi\eta),\ i = \overline{1,n}. \end{aligned} \tag{6}$$

Whence for convex interpolation with cubic C^1 splines, the functions $F_i^*,\ i=\overline{1,n}$, and the sets $W_i,\ i=\overline{1,n}$, in (D_{FL}^{ts}) are defined by (6) and (5), respectively. Moreover, as

$$\nabla F_i(x,y)=\frac{4}{h_i^2}(2x+y-3\tau_i,x+2y-3\tau_i)^T,\ i=\overline{1,n},$$

in this case (2) becomes

$$\begin{aligned}0\in(-q_{i-1},q_i)^T &+ \frac{4}{h_i^2}(2u_{i-1}+u_i-3\tau_i,u_{i-1}+2u_i-3\tau_i)^T\\ &+ N_{W_i}(u_{i-1},u_i),\ i=\overline{1,n},\end{aligned} \tag{7}$$

where $q=(q_0,q_1,...,q_{n-1},q_n)^T\in\mathbb{R}^{n+1}$ with $q_0=q_n=0$ and $u=(u_0,...,u_n)^T\in\mathbb{R}^{n+1}$.

In order to establish an algorithm for constructing convex interpolation with cubic C^1 splines, let us rewrite (7) as follows:

$$0\in A_i(q_{i-1},q_i,u_{i-1},u_i,\tau_i)+B_i(u_{i-1},u_i),\quad i=1,...,n, \tag{8}$$

where τ_i are given and $q_i,\ u_i$ are unknown variables and

$$\begin{aligned}A_i(q_{i-1},q_i,u_{i-1},u_i,\tau_i):&=\Big(\frac{4}{h_i^2}(2u_{i-1}+u_i-3\tau_i)-q_{i-1},\frac{4}{h_i^2}(u_{i-1}+2u_i-3\tau_i)+q_i\Big)^T,\\ B_i(u_{i-1},u_i):&=N_{W_i}(u_{i-1},u_i).\end{aligned}$$

Algorithm *ConvSpline*

Input: $q^0,\ q_0^1,\ q_0^2\in\mathbb{R},z_0=(z_0^1,z_0^2)^T\in\mathbb{R}^2$,

Output: A solution of the problem (8): $(u_0,...,u_n)^T\in\mathbb{R}^{n+1}$.

Begin

Step 1. Choose $i:=1,\ \ q^0:=0,\ \ q_0^1,q_0^2\in[-L,+L],\ q_0^1\neq q_0^2,\ L>0$ and $\varepsilon>0$.

(i) Setting $\hat{q}_0 := (q^0, q_0^1)^T, \; \tilde{q}_0 := (q^0, q_0^2)^T$, for i compute forward step

$$s_i^1 := (I - \lambda A_i(\hat{q}_0, z_0, \tau_i))$$

$$s_i^2 := (I - \lambda A_i(\tilde{q}_0, z_0, \tau_i)).$$

(ii) Solve backward step by projection method that is a quadratic programming problem.

$$z_i^1 := P_{W_i}(s_i^1) \in \mathbb{R}^2$$

$$z_i^2 := P_{W_i}(s_i^2) \in \mathbb{R}^2.$$

(iii) Stop criteria: if $\| z_i^1 - z_i^2 \| < \varepsilon$, then $\quad z_i^2 :\approx (u_{i-1}, u_i)^T$.

Step 2. Update q_0 :

$$\begin{aligned} \bar{q}_0 : &= \hat{q}_0 - \alpha \tilde{q}_0, \; \alpha > 0, \; \bar{q}_0 \in [-L, +L], \\ z_i^1 : &= z_i^2, \\ \tilde{q}_0 : &= \bar{q}_0. \end{aligned}$$

Goto Step 1(i)

Step 3.

$$\begin{aligned} i : &= i + 1, \\ q^0 : &= -q_0^2, \\ z_0 : &= z_{i-1}^2. \end{aligned}$$

Goto Step 1(i)

End.

Remark. The problem (8) is actually a system of inclusion problems depending on q_i. In order to choose parameters q_{i-1}, q_i, we apply a descent direction to Algorithm *ConvSpline.* Depending on the choice parameters in (8), can be considered some alternative versions of the Algorithm *ConvSpline*.

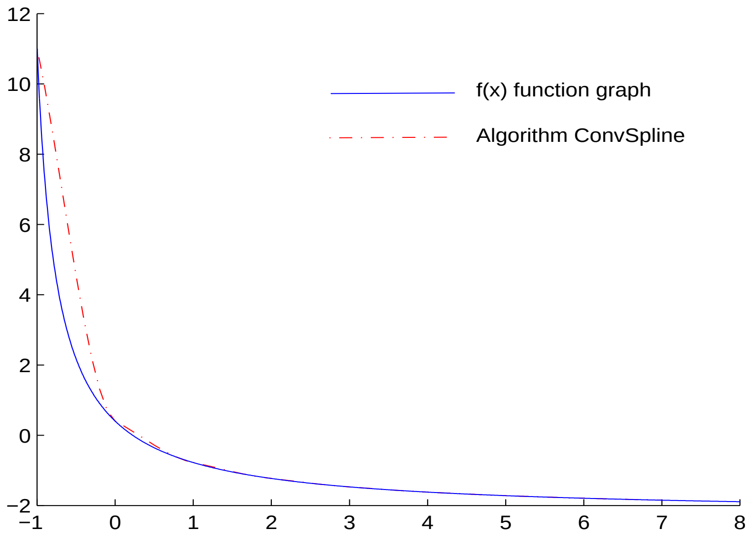
Example 1. ([8], Fiorot and Tabka)
We interpolate ten points of the graph of a given function

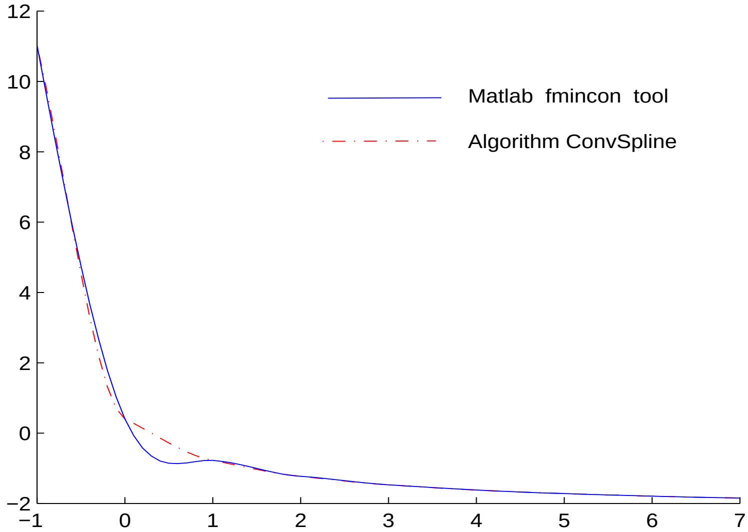
$$f(x) = \frac{-9x+2}{4x+5},$$

which is decreasing and convex on the interval $[-1, 8]$. We use a uniform subdivision $h_i = 1$ for $i = 1, ..., 8$ and values of the function at points $x_i = i,\ i = -1, 0, 1, 2, 3, 4, 5, 6, 7, 8$. The graph of the function can be compared to the results computed by Algorithm *ConvSpline* and by Matlab cubic spline tools. Numerical results were tested by using Matlab tools on a Toshiba L305D 2.0GHz processor with 3.0 GB RAM.

Table 1.

n/h	Matlab fmincon tools	Algorithm *ConvSpline*	Iteration	Time(Sec)	Relative error
n=9,h=1	-13.1461	-10.6000	2	0.0277	0.1937
	-5.5077	-1.1778	3	0.0271	0.7862
	-0.1563	-0.4530	2	0.0232	1.8983
	-0.1563	-0.2398	1	0.0319	0.5342
	-0.1563	-0.1485	4	0.0401	0.0499
	-0.1273	-0.1010	3	0.0181	0.2066
	-0.0828	-0.0731	2	0.0228	0.1171
	-0.0637	-0.0554	2	0.0269	0.1303
	-0.0479	-0.0434	2	0.0265	0.0939

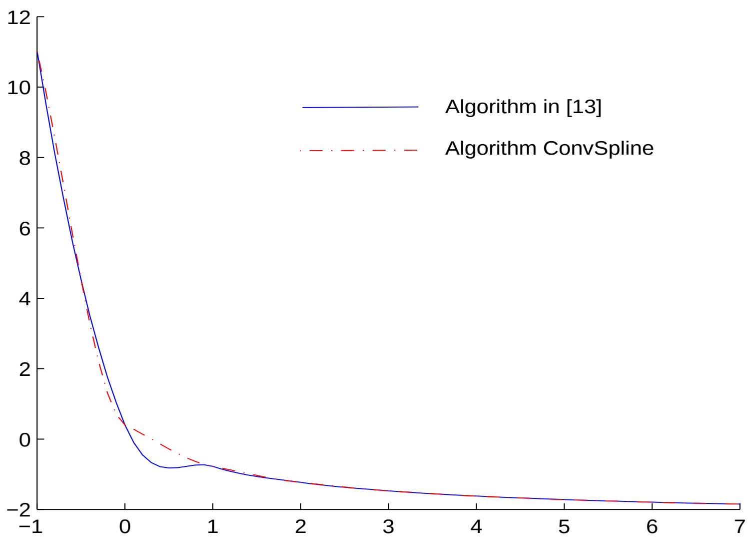




The next table is devoted to the results comparing the Algorithm *ConvSpline* to the Algorithm in [13].

Table 2.

n/h	Algorithm in [13]	Algorithm *ConvSpline*	Iteration	Time(Sec)	Relative error
n=9,h=1	-15.3111	-10.6000	4	0.0277	0.3077
	-5.8889	-1.1778	3	0.0271	0.8000
	-0.8154	-0.4530	2	0.0232	0.4444
	-0.3464	-0.2398	5	0.0319	0.3077
	-0.1941	-0.1485	2	0.0401	0.2349
	-0.1247	-0.1010	3	0.0181	0.1901
	-0.0870	-0.0731	2	0.0228	0.1598
	-0.0642	-0.0554	1	0.0269	0.1371
	-0.0465	-0.0434	2	0.0265	0.0667

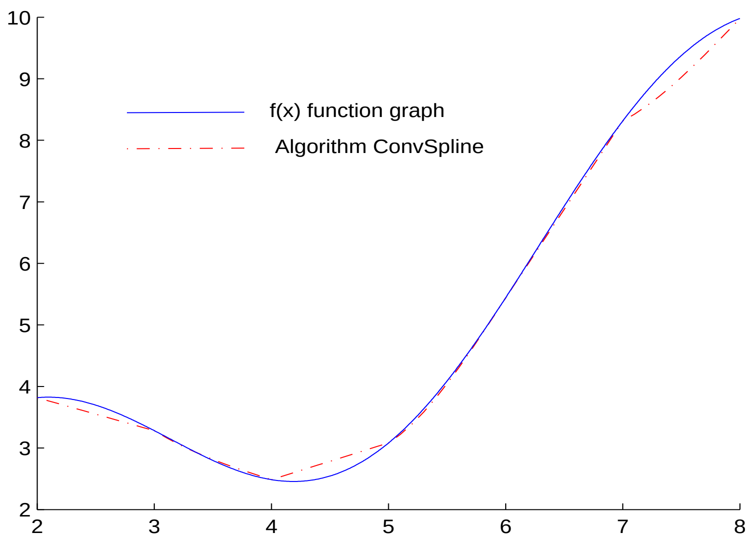


Example 2. Now we consider the following function which is increasing and convex on the interval $(4, 7)$.

$$f(t) = 2\sin x + x.$$

Table 3.

n/h	Matlab fmincon tools	Algorithm *ConSpline*	Iteration	Time(Sec)	Relative error
	-0.6510	-0.5364	5	0.0801	0.1760
	-0.6510	-0.4953	2	0.0630	0.2392
	-0.3973	-0.5958	4	0.0254	0.4996
	-1.6399	-1.3820	6	0.0249	0.1573
n=7,h=1	-1.6399	-2.8475	3	0.0238	0.7364
	-1.6399	-2.8728	3	0.0266	0.7518
	-1.6399	-2.1854	2	0.0228	0.3326



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